Journal of Statistical Physics, Vol. 124, No. 6, September 2006 (© 2006)
DOI: 10.1007/s10955-006-9192-x

Estimating the Solutions of the Boltzmann Equation

Carlo Cercignani

Received July 13, 2006; accepted August 1, 2006
Published Online: September 6, 2006

We discuss some possible estimates of the solutions of the Boltzmann equation, which might permit a progress in the theory of existence of weak solutions.

KEY WORDS: Boltzmann equation, weak solutions, global solutions

1. INTRODUCTION

It seems rather clear that in order to achieve some progress in the study of the initial value problem for the nonlinear Boltzmann equation and prove that the typical solutions have more properties that those proved in the well-known theorem by DiPerna and Lions,[(1)] more *a priori* estimates are needed. When the solution depends on just one space coordinate, these are available as shown in a previous paper,[(2)] where the solution is shown to be a weak one without any need for renormalization with the consequence that conservation of energy holds.

In this paper we shall illustrate a technique leading to new estimates. For the sake of generality we shall consider the Boltzmann equation[(3)] in R^n

$$\partial_t f + \xi \cdot \partial_x f = Q(f, f) \tag{1.1}$$

where $Q(f, f)(x, \xi, t)$ is the quadratic collision term. We shall not need the explicit form of the collision term, but only its property expressed by the conservation laws. In fact, we want to obtain estimates for the quadratic collision integral; a way to make progress in this direction seems to be offered by a study of the quadratic estimates following from the linear conservation equations that do not depend on the details of the right hand side of the equation.

The aim of this paper is to provide some simple results which might turn out to be a prerequisite for further progress in the field. The plan of the paper is as follows: Sec. 2 contains three basic lemmas, which, in the author's opinion,

[1] Dipartimento di Matematica Politecnico di Milano, Milano, Italy.

should be the starting point of a possible progress; Sec. 3 discusses a possible exploitation of the lemmas by an interesting connection between the theory of the Boltzmann equation and the theory of Newtonian potential, which leads to the desired estimate, unfortunately, just in the case of a solution depending on a space coordinate, when the matter has been previously clarified [Ce][2]; Sec. 4 considers a possible role of the balance of angular momentum, which is meaningless in the case of one-dimensional solutions; Sec. 5 contains a few concluding remarks.

In order to appreciate the spirit of the present paper, the reader should keep in mind that the author has avoided following the style of Gauss, who, according to Abel, 'is like the fox, who effaces his tracks in the sand with his tail'. Rather, we have tried to make the presentation as naive as possible, in order to stimulate further thinking about the subject.

2. THE BASIC LEMMAS

In this section we want to study a few estimates for the solutions of the Boltzmann equation. We introduce the notation $d\mu = d\xi d\eta dx dy$ and start with the following.

Lemma 1. *Let $f(x, \xi, t)$ be a (renormalized) solution of the Boltzmann equation in R^n depending on m space variables $(m \leq n)$. Assume the integral $\int f(x, \xi, t) f(y, \eta, t) g(x - y) d\mu$ exists for some function g and any $t \in R_+$. Then the integral $\int_0^t \int f(y, \xi, t') f(x, \eta, t')(\xi - \eta) \cdot \partial_x g(x - y) d\mu dt'$ also exists. The derivative is to be considered distributionally if needed.*

To prove the lemma, we remark that, because of mass conservation[(3)]:

$$\int \xi \cdot \partial_x f(x, \xi, t) d\xi = -\int \partial_t f(x, \xi, t) d\xi \tag{2.1}$$

and changing x in to y ξ into η:

$$\int \eta \cdot \partial_y f(y, \eta, t) d\eta = -\int \partial_t f(y, \eta, t) d\eta \tag{2.2}$$

Let us first multiply the first equation by $f(y, \eta, t) g(x - y)$ and integrate the result with respect to η, x, y, t and add to this the result of the same operation on the second equation (with ξ in place of η and x in place of y). We obtain by partially integrating in the left hand side

$$\int_0^t \int f(y, \eta, t') f(x, \xi, t')(\xi - \eta) \cdot \partial_x g(x - y) d\mu dt'$$
$$= \int f(x, \xi, t) f(y, \eta, t) g(x - t) d\mu - \int f(x, \xi, 0) f(y, \eta, 0) g(x - y) d\mu \tag{2.3}$$

and the result follows.

We now want to prove

Lemma 2. *Let $f(x, \eta, t)$ be a (renormalized) solution of the Boltzmann equation in R^n depending on m space variables ($m \leq n$). Assume the integral $\int f(x, \xi, t) f(y, \eta, t)(\xi - \eta) \cdot v(x - y) d\mu$ to be bounded for the some function vector-valued v and any $t \in R_+$. Then the integral $\int_0^t \int f(x, \xi, t') f(y, \eta, t')(\xi - \eta) \cdot \partial_x[(\xi - \eta) \cdot v(x - y)] d\mu dt'$ also exists. The derivative is to be considered distributionally if needed.*

The proof of this lemma is similar to the previous one. We take the momentum conservation [(3)] minus mass conservation multiplied by η:

$$\int (\xi - \eta)\xi \cdot \partial_x f(x, \xi, t) d\xi = -\int (\xi - \eta)\partial_t f(x, \xi, t) d\xi \tag{2.4}$$

and changing x in y, ξ into η:

$$\int (\eta - \xi)\eta \cdot \partial_y f(x, \eta, t) d\eta = -\int (\eta - \xi)\partial_t f(y, \eta, t) d\eta \cdot \tag{2.5}$$

Let us scalarly multiply the first equation by $f(y, \eta, t)v(x - y)$ and integrate the result with respect to η and y and add to this the result of the same operation on the second equation (with ξ in place of η and x in place of y). We obtain by partially integrating in the left hand side

$$\begin{aligned}
&\int_0^t \int f(y, \eta, t') f(x, \xi, t')(\xi - \eta) \cdot \partial_x[(\xi - \eta) \cdot v(x - y)] d\mu dt' \\
&\quad = \int f(x, \xi, t) f(y, \eta, t)(\xi - \eta) \cdot v(x - y) d\mu \\
&\quad - \int f(x, \xi, 0) f(y, \eta, 0)(\xi - \eta) \cdot v(x - y) d\mu
\end{aligned} \tag{2.6}$$

and the result follows.

We can also combine the two lemmas to yield

Lemma 3. *Let $f(x, \xi, t)$ be a (renormalized) solution of the Boltzmann equation in R^n depending on m space variables ($m \leq n$). Assume the integral $\int f(x, \xi, t) f(y, \eta, t) g(x - y) d\mu$ exists for some function g and any $t \in R_+$. Then the integral $\int_0^t (t - t') f(x, \xi, t') f(y, \eta, t')(\xi - \eta) \cdot \partial_x[(\xi - \eta) \cdot \partial_x g(x - y)] d\mu dt'$ exists finite. The derivative is to be considered distributionally if needed.*

To prove the lemma it is sufficient to apply Lemmas 1 and 2 with $v = \partial_x g(x-y)$. This yields the existence of

$$\int_0^t \int_0^{t'} \int f(x,\xi,t')f(y,\eta,t')(\xi-\eta)\cdot\partial_x[(\xi-\eta)\cdot\partial_x g(x-y)]d\mu dt'\,dt''$$
$$= \int_0^t \int (t-t')f(x,\xi,t')f(y-\eta,t')(\xi-\eta)\cdot\partial_x[(\xi-\eta)\cdot\partial_x g(x-y)]d\mu dt' \tag{2.7}$$

3. HOW TO EXPLOIT THE LEMMAS

In this section we examine the role that the lemmas proved in the previous section may have in order to obtain useful estimates. One can start from, say, a bounded kernel with singular derivatives and end up with an estimate involving a more singular kernel. The ideal result would be to end up with a delta function because then we would obtain a significant estimate for the collision term. This is beautifully exemplified by solutions depending on just one space variable ($m = 1$); by taking $g = |x-y|$ we have an estimate of the kind we are looking for. This kind of result was exploited in the previous paper.

Things prove much harder in more dimensions ($m > 1$). An obvious idea would be to take $g = g_m$, where g_m is the usual Newtonian potential ($= (2-m)^{-1}\,|x-y|^{2-m}$, if $m \neq 2$; $= \log|x-y|$ if $m = 2$). Then if $\rho = \int f\,dv$ and $\int \rho(x,t)\rho(y,t)g_m(|x-y|)$ exist finite, then the collision term is (weakly) in L^1, and f is a weak solution. (The case $m = 1$ is the case alluded to before).

Let us see what happens for $m > 1$. Let us denote the components of x and ξ by x_i anf ξ_i. The kernel $K_{ij} = \partial_{xi}\partial_{x_j} g_m(x-y)(m > 1)$ is made up of a Dirac delta and a distribution of the kind "principal part" (for $m = 1$ this second part is absent).

To be precise let us denote by $\tilde{K}_{ij}$ the pointwise derivatives. defined for $x \neq y$:

$$\tilde{K}_{ij}(x-y) = [x-y]^{-m}\left(\delta_{ij} - m\frac{(x-y)_i(x-y)_j}{|x-y|^2}\right), \tag{3.1}$$

where δ_{ij} the Kronecker symbol. The following equality holds:

$$\int K_{ij}(x-y)\phi(x)dx = \mathrm{P}\int \phi(x)\tilde{K}_{ij}(x-y)dx + \omega_m\phi(y)$$
$$= \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} \phi(x)\tilde{K}_{ij}(x-y)dx + \omega_m\phi(y) \tag{3.2}$$

where ω_m is the area of the unit sphere in m dimensions and P in front of an integral denotes that the Cauchy principal part must be considered.

Using the lemmas of the previous sections we can prove that, if the integral $\int \rho(x,t)\rho(y,t)g_m(|x-y|)dxdy$ exists finite, then

$$\int_0^t \int (t-t')f(x,\xi)f(y,\eta)(\xi_i-\eta_i)(\xi_j-\eta_j)K_{ij}(x-y)d\mu dt' \tag{3.3}$$

(sum over i and j from 1 to m) exists finite. But the question is: does also

$$\int_0^t \int (t-t')f(x,\xi)f(x,\eta)|\xi-\eta|^2 dxd\xi d\eta dt' \tag{3.4}$$

exist finite? (Remark: the result is trivial for $m = 1$). A positive answer seems unlikely for $m > 1$. In this case we must enquire about other possible estimates. A possibility might be offered by the balance of angular momentum; this will be discussed in the next section.

Before ending this section, however, we must remark that even the assumption that the integral $\int \rho(x,t)\rho(y,t)g_m(|x-y|)dxdy$ exists finite is far from trivial, though more innocent that the analogous assumption on

$$\int_0^t \int (t-t')f(x,\xi)f(y,\eta)(\xi_i-\eta_i)(\xi_j-\eta_j)K_{ij}(x-y)d\mu dt'. \tag{3.5}$$

We just remark that a solution of the Boltzmann equation is certainly more regular than usually assumed. Without resorting to other considerations and just using the material of this paper, e.g. Lemma 3 with $K(x-y) = |x_1-y_1|$, we can prove that

$$\int_0^l \int (t-t')f(x_1,x_2,\ldots,x_m,\xi)$$
$$f(x_1,y_2,\ldots,y_m,\eta)(\xi_1-\eta_1)^2 dy_2\ldots dx_m d\xi d\eta dt'$$

where it is to be remarked that the first argument is the same in the two functions and the integral is with respect to $2m-1$ rather than $2m$ space coordinates.

4. ANGULAR MOMENTUM

The balance of angular momentum follows from the Boltzmann equation in the following form:

$$\int \partial_t(x_i\xi_k - x_k\xi_i)f(x,\xi,t)d\xi = -\int \xi\cdot\partial_x(x_i\xi_k - x_k\xi_i))f(x,\xi,t)d\xi \tag{4.1}$$

For later use it is convenient to introduce two other vectors y and η and rewrite this equation in the following, more cumbersome form:

$$\int \partial_t[(x_i - y_i)(\xi_k - \eta_k) - (x_k - y_k)(\xi_i - \eta_i)]f(x,\xi,t)d\xi$$
$$= -\int \xi \cdot \partial_x[(x_i - y_i)(\xi_k - \eta_k) - (x_k - y_k)(\xi_i - \eta_i)]f(x,\xi,t)d\xi$$
$$- \int (\xi_i\eta_k - \xi_k\eta_i)f(x,\xi,t)d\xi \tag{4.2}$$

and, interchanging x and y, ξ and η:

$$\int \partial_t[(x_i - y_i)(\xi_k - \eta_k) - (x_k - y_k)(\xi_i - \eta_i)]f(x,\eta,t)d\eta$$
$$= -\int \eta \cdot \partial_x[(x_i - y_i)(\xi_k - \eta_k) - (x_k - y_k)(\xi_i - \eta_i)]f(x,\eta,t)d\eta$$
$$- \int (\xi_k\eta_i - \xi_i\eta_k)f(x,\xi,t)d\xi \tag{4.3}$$

Now, let $H(|x-y|)$ be another kernel. If we multiply the first equation by $x_i y_k f(y,\eta,t)H(|x-y|)$ and integrate the result with respect to η and y, the second by $x_i y_k f(x,\xi,t)H(|x-y|)$ and integrate the result with respect to ξ and x (sum from 1 to m on the indices i and k), and finally add the results, we obtain:

$$\int [x \cdot (x-y)y \cdot (\xi - \eta) - y \cdot (x-y)x \cdot (\xi - \eta)]$$
$$f(x,\xi,t)f(x,\xi,t)H(|x-y|)d\mu$$
$$- \int [x \cdot (x-y)y \cdot (\xi - \eta) - y \cdot (x-y)x \cdot (\xi - \eta)]f(x,\xi,0)$$
$$f(x,\eta,0)H(|x-y|)d\mu$$
$$= \int [x \cdot (x-y)y \cdot (\xi - \eta) - y \cdot (x-y)x \cdot (\xi - \eta)]f(x,\xi,t')$$
$$f(x,\xi,t')(\xi - \eta) \cdot \partial_x H(|x-y|)d\mu dt' \tag{4.4}$$

because

$$-(x_i\xi_i y_k\eta_k - y_k\xi_k x_i\eta_i) - (y_k\xi_k x_i\eta_i - x_i\xi_i y_k\eta_k) = 0$$

Let us remark that the terms arising from the differentiation of the factor $x_i y_k$ turn out to cancel because they contain the factor

$$y \cdot (\xi - \eta) + (x-y) \cdot \xi - (x-y) \cdot \eta - (\xi - \eta) \cdot x$$
$$= -(x-y) \cdot (\xi - \eta) + (x-y) \cdot (\xi - \eta) = 0 \tag{4.5}$$

Let us also remark that this is one of infinitely many relations that we might obtain. In fact, if a is an arbitrary constant vector, instead of multiplying the first equation by $x_i y_k f(x,\eta,t)H(|x-y|)$, the second by $x_i y_k f(x,\xi,t)H(|x-y|)$, we might replace the factor $x_i y_k$ by $(x_i + a_i)(y_k + a_k)$. If we do this, the factor in square brackets contains the following additional terms:

$$\begin{aligned} &a\cdot(x-y)y\cdot(\xi-\eta) - a\cdot(x-y)x\cdot(\xi-\eta) + a\cdot(x-y)a\cdot(\xi-\eta)\\ &-a\cdot(x-y)a\cdot(\xi-\eta) = a\cdot(x-y)y\cdot(\xi-\eta)\\ &-a\cdot(x-y)x\cdot(\xi-\eta) = -a\cdot(x-y)(x-y)\cdot(\xi-\eta). \end{aligned} \tag{4.6}$$

Thus, since a is arbitrary, the following relation must also be true:

$$\begin{aligned} &\int (x-y)(x-y)\cdot(\xi-\eta)f(x,\xi,t)f(x,\eta,t)H(|x-y|)d\mu\\ &\quad - \int (x-y)(x-y)\cdot(\xi-\eta)f(x,\xi,0)f(x,\eta,0)H(|x-y|)d\mu\\ &\quad = \int (x-y)(x-y)\cdot(\xi-\eta)f(x,\xi,t')\\ &\qquad\quad f(x,\xi,t')(\xi-\eta)\cdot\partial_x H(|x-y|)d\mu dt' \end{aligned} \tag{4.7}$$

However, it is by no means evident that these further relations may help in answering the question raised in the previous section.

5. CONCLUDING REMARKS

We have discussed a method to obtain new estimates for the solutions of the Boltzmann equation. Although the progress achieved is not so great, the method recommends itself because it produces the desired result when the solution depends on just one space coordinate. Hopefully, some further ideas or some progress on the technical question raised in Sec. 3 will lead to a significant new result.

ACKNOWLEDGMENTS

The research described in this paper was supported by MURST of Italy.

REFERENCES

1. R. DiPerna and P. L. Lions, On the Cauchy problem for Boltzmann equations: Global existence and weak Stability. *Ann. Math.* **130**:321–366 (1989).
2. C. Cercignani, Global weak solutions of the Boltzmann Equation. *J. Stat. Phys.* **118**:333–342 (2005).
3. C. Cercignani, R. Illner and M. Pulvirenti, *The Mathematical Theory of Dilute Gases* (Springer-Verlang, New York, 1994).